Subject: Physics

Production of Courseware

-Content for Post Graduate Courses

Paper No. : Electromagnetic Theory

Module : Maxwell's Equations - I

Development Team

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Description of Module	
Subject Name	Physics
Paper Name	Electromagnetic Theory
Module Name/Title	Maxwell's Equations - I
Module Id	

Contents

Learning Objectives:

From this module students may get to know about the following:

1. The modification in the equations of electricity and magnetism for time-varying fields.
2. Maxwell's equations in vacuum and the modifications needed in a medium.
3. Introduction of potentials – the first step towards the solution of the equations.

1. Introduction

We have studied the subject of electricity and magnetism in great detail in our undergraduate classes. The basic laws of electricity and magnetism which you learnt in the undergraduate course are summarized in the following equations:

Coulomb's law $\vec{\nabla}.\vec{E} = \rho / \varepsilon_0$.

Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Faraday's law: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

Absence of free magnetic poles $\vec{\nabla}.\vec{B} = 0$

These equations obviously need some elaboration. First of all these quantities, $\vec{E}, \vec{B}, \vec{J}$ and ρ are fields, i.e., they are functions of space and time. $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ are the electric field and magnetic induction respectively at the point $\vec{x}$ at time t. ρ and $\vec{J}$ are respectively the charge and current densities at ($\vec{x}$,t).

These are the differential forms of the equations of electricity and magnetism. You have also studied all these equations in their integral forms as well.

2. Maxwell's equations

2.1 Displacement Current:

If you remember, in the undergraduate course these laws were derived from steady-state observations. Consequently it is quite possible that these equations may (and indeed do) need modifications in the presence of time-varying fields. In fact, the equations as they stand are inconsistent. The problem is with the Ampere's law. If any physical quantity is conserved, the rate at which it increases within a volume must equal the rate at which it enters into the volume through its boundary. This conservation law is expressed in terms of an equation of continuity. In the case of conservation of charge, it takes the form

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (1)$$

Now for steady-state phenomena, $\frac{\partial \rho}{\partial t} = 0$, and the equation reduces to $\vec{\nabla}.\vec{J} = 0$, and you can see that it is consistent with the Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \vec{\nabla}.(\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla}.\vec{J}$$

The left hand side of this equation is zero, since the divergence of the curl of a vector field is zero. For steady state the right hand side is also zero. Hence there is no inconsistency. However it is clear that for time varying fields, whereas the left hand side of the above equation is zero, the right hand side is not. The way out was found by Maxwell who realized that by using the Coulomb's law the continuity equation could be rewritten in an alternative form. Using Coulomb's law $\vec{\nabla}.\vec{E} = \rho / \varepsilon_0$, the continuity equation

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

takes the form

$$\vec{\nabla}.(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = 0.$$

Maxwell now replaced $\vec{J}$ in Ampere's law by

$$\vec{J} \rightarrow \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

for time-dependent fields. The Ampere's law thus becomes

$$\vec{\nabla}\times\vec{B}=\mu_0(\vec{J}+\varepsilon_0\partial\vec{E}/\partial t)\,.$$

For steady currents the law remains the same. However for time-varying fields now there is an additional term, $\varepsilon_0\partial\vec{E}/\partial t$, called the displacement current.

2.2 Maxwell's equations in vacuum

This seemingly small addition to the Ampere's law had far-reaching consequences. Without it there would not be any electromagnetic waves and of course no connection with optics. The set of four equations

$$\vec{\nabla}.\vec{E}=\rho/\varepsilon_0\,, \tag{2}$$

$$\vec{\nabla}\times\vec{B}=\mu_0(\vec{J}+\varepsilon_0\partial\vec{E}/\partial t) \tag{3}$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0\,, \tag{4}$$

$$\vec{\nabla}.\vec{B}=0 \tag{5}$$

are together known as the Maxwell equations. When combined with the Lorentz force equation

$$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B}) \tag{6}$$

which is the force on a particle with charge q moving with velocity $\vec{v}$, and the Newton's law

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{7}$$

provides a complete description of all phenomena involving matter and fields. The set of four equations (2 – 5) as they have been written above apply to electromagnetic phenomena in vacuum.

2.2 Maxwell's Equations in Matter:

Maxwell's equations in the above form are complete and correct as they stand. They are valid in vacuum as well as in materials. However as we know from study of electrostatics and magnetostatics, when you are working with materials, they are subject to electric and magnetic polarization leading to bound charges and currents. We would like that our equations involve only free charges and currents which we supply rather than the sum of the free and bound quantities.

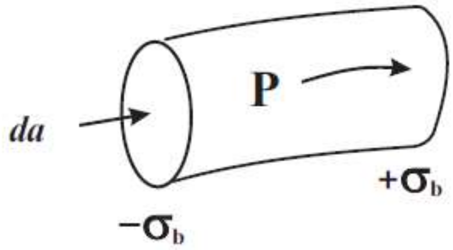
We already know from the static case that an electric polarization $\vec{P}$ produces a bound charge density

$$\rho_b = -\vec{\nabla}.\vec{P} \tag{8}$$

Similarly, a magnetization $\vec{M}$ results in a bound current

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \tag{9}$$

The new element is that any change in the polarization charge induces a bound current which must also be included in the total current. Consider a small volume element of the material. **[See Figure 7.45– Griffiths p. 329]**



Polarization introduces a charge density $\sigma_b = P$ at one end of the material and $-\sigma_b$ at the other. If now P increases a bit, charge on each end increases leading to a net current:

$$dI = \frac{d\sigma_b}{dt} da_\perp = \frac{dP}{dt} da_\perp$$

This produces a current density

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t} \qquad (10)$$

Point to note: Please note that the polarization current $\vec{J}_P$ is independent of the bound current $\bar{J}_b$. Whereas $\bar{J}_b$ is due to magnetization and is associated with orbital and spin angular momentum of the electron, $\vec{J}_P$ is a result of linear motion of the electrons due to changing polarization. The polarization current is consistent with continuity equation; in fact it is required for the conservation of bound charge:

$$\vec{\nabla}.\vec{J}_P = \vec{\nabla}.\frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla}.\vec{P}) = -\frac{\partial \rho_b}{\partial t}$$

Thus the total charge density can be written as a sum of two parts, the free charge density and the bound charge density:

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}.\vec{P} \tag{11}$$

and the total current density as a sum of three parts, the free, the bound and the polarization current density:

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \tag{12}$$

Using these, the Gauss law can be rewritten as

$$\vec{\nabla}.\vec{E} = \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla}.\vec{P})$$

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$$\vec{\nabla}.\vec{D} = \rho_f \tag{13}$$

where $\vec{D}$, the displacement vector is given by

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{14}$$

The Ampere's law (with Maxwell's modification) becomes

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$= \mu_0(\vec{J}_f + \vec{\nabla}\times\vec{M}) + \mu_0\frac{\partial}{\partial t}(\varepsilon_0\vec{E} + \vec{P})$$

$$= \mu_0(\vec{J}_f + \vec{\nabla}\times\vec{M}) + \mu_0\frac{\partial\vec{D}}{\partial t}$$

or

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \qquad (15)$$

where

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad (16)$$

The other two Maxwell equations are not affected since they do not involve density and current. The vectors $\vec{D}$ and $\vec{H}$ are two auxiliary fields that are introduced to absorb the effects of polarization and magnetization, so that Maxwell equations can be written in terms of free or external charges and currents only.

In terms of the free charge density and current density, Maxwell's equations in a material read

$$\vec{\nabla}.\vec{D} = \rho_f\,; \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}\,; \qquad (17)$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0\,; \qquad \vec{\nabla}.\vec{B} = 0 \qquad (18)$$

Some physicists regard these as the more fundamental set of equations. To some extent this is a matter of choice. What we have done is to conveniently divide the total charge and current into free and bound parts. That does not make the equation more fundamental; only more convenient practically. The disadvantage is that two more auxiliary relations have to be introduced. These are the constitutive relations which give $\vec{D}$ and $\vec{H}$ in terms of $\vec{E}$ and $\vec{B}$, respectively. These relations depend on the nature of the materials.

Most materials are linear (particularly in the weak field limit); for them the constitutive relations are

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{19}$$

$$\vec{M} = \chi_m \vec{H} \tag{20}$$

Substituting for $\vec{P}$ into equation (14) we get

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi\ )\vec{E}$$

or

$$\vec{D} = \varepsilon \vec{E}\,, \tag{21}$$

where

$$\varepsilon = \varepsilon_0 (1 + \chi_e) \tag{22}$$

The constant χ_e is called the electric susceptibility of the material, and the constant ε is called the permittivity of the material. This expression is exactly the same as in the static case. The quantity

$$\kappa = (1+\chi_e) = \varepsilon/\varepsilon_0 \qquad (23)$$

is called the dielectric constant of the material.

In a similar manner substituting for $\vec{M}$ from equation (20) into (16), we have

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H}$$

or

$$\vec{B} = \mu\vec{H}$$

where

$$\mu = \mu_0(1+\chi_m) = \mu_0\mu_r; \qquad \mu_r = (1+\chi_m)$$

The constant χ_m is called the magnetic susceptibility of the material and μ is called the permeability of the material. The constant μ_r may be called relative permeability. For free space there is neither any polarization nor any magnetization, so $\varepsilon = \varepsilon_0; \mu = \mu_0$. Therefore, ε_0 is called the permittivity of free space and μ_0 the permeability of free space.

For linear materials, the change is minimal; Maxwell's equations remain same in form, the constants ε_0 and μ_0 are replaced by ε and μ respectively and the sources are to be considered as free sources:

$$\vec{\nabla}.\vec{E} = \rho/\varepsilon\,, \qquad (2)$$

$$\vec{\nabla} \times \vec{B} = \mu(\vec{J} + \varepsilon \partial \vec{E} / \partial t) \quad (3)$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \, , \quad (4)$$

$$\vec{\nabla} . \vec{B} = 0 \quad (5)$$

3. Scalar and Vector Potentials

Now that we have the Maxwell's equations – the basic equations of electrodynamics, our next aim should be to try to solve them. Equations of electrostatics ($\vec{B}$ = 0, $\vec{E}$ static), magnetostatics and steady currents are all special cases of Maxwell's equations which we have solved, at least for some simple configurations in the undergraduate course. Our aim now is to solve them in their full glory. How do we go about this task? These are a set of four coupled partial differential equations, two scalar and two vector, for the vector fields $\vec{E}$ and $\vec{B}$. These have to be solved for various possible configurations and for various types of initial and boundary conditions. **This is as complicated as it could possibly get.**

The first step in this direction is the introduction of scalar and vector potentials, in analogy with the case of electrostatics (and magnetostatics), where we introduce the scalar potential Φ, and the Coulomb's law turns into Poisson equation for Φ. We will do some thing similar here also. We start with the homogenous equation $\vec{\nabla} . \vec{B} = 0$. Now we know from vector analysis that any divergenceless vector field can be written as the curl of another vector field. Thus if we put

$$\vec{B} = \vec{\nabla} \times \vec{A} \, , \quad (19)$$

then equation (5) is automatically satisfied. $\vec{A}$ is called the vector potential.

Now look at the other homogenous equation [equation (4)]:

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0\,.$$

Using Equation (8), this takes the form

$$\vec{\nabla}\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$$

which means that $(\vec{E}+\frac{\partial\vec{A}}{\partial t})$ must be the gradient of a scalar field, called the scalar potential which is traditionally taken to be $-\Phi(\vec{x},t)$:

$$(\vec{E}+\frac{\partial\vec{A}}{\partial t})=-\vec{\nabla}\Phi\,,$$

or

$$\vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}\,. \qquad (20)$$

What it all means is that if you choose any arbitrary scalar field $\Phi(\vec{x},t)$ and any vector field $\vec{A}(\vec{x},t)$, define the electric field $\vec{E}(\vec{x},t)$ and the magnetic induction $\vec{B}(\vec{x},t)$ via equations (16) and (15) respectively, then the two homogenous Maxwell equations are satisfied automatically. The dynamical behaviour of the fields is determined by the two inhomogeneous equations which involve the sources of the fields: $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$.

Since we are looking at the microscopic form of the equations (15-18), using the definitions of $\vec{E}$ and $\vec{B}$ in terms of the potentials, equations (19) and (20), in equations (15) and (16), gives

$$\vec{\nabla}.\vec{E} = \vec{\nabla}(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}) = \rho / \varepsilon_0 \, ,$$

or

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\rho / \varepsilon_0 \, . \qquad (21)$$

Using the definition of $\vec{E}$ and $\vec{B}$ in the second inhomogeneous equation, the left side becomes

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 \vec{A} \, ,$$

where the second equality is the well known result for triple vector product. Similarly, the right hand side of the equation becomes

$$\mu_0 (\vec{J} + \varepsilon_0 \partial \vec{E} / \partial t) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t})$$

On equating the two sides and rearranging the terms, we obtain

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla}.\vec{A} + \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t}) = -\mu_0 \vec{J} \, . \qquad (22)$$

Now the quantity $(\mu_0 \varepsilon_0)^{-1/2}$ has the dimensions of velocity and numerically it equals the velocity of light in vacuum. Let us, therefore, replace $(\mu_0 \varepsilon_0)^{-1/2}$ by c, the velocity of light in vacuum and rewrite the above equation as

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}.(\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}) = -\mu_0\vec{J} \qquad (22a)$$

We have reduced the set of four Maxwell's equations into a set of two equations. However in a sense we have not achieved much. Instead of being first order they are now second order differential equations, and more importantly, they are still coupled. What is important is to somehow decouple the equations so that they can be solved independently. That is the next step in our development of the subject which we will take up in the next module.

Summary

1. It was explained that Ampere's law as it stands is inconsistent with the equation of continuity which expresses the law of conservation of charge.

2. Next, how Maxwell modified Ampere's law for time-varying phenomena to make it consistent with continuity equation was explained.

3. Maxwell's equations were written in complete generality in vacuum as well as in a medium by taking polarization and magnetization into account.

4. The concept of scalar and vector potentials was introduced. The introduction of potentials simplifies the equations to some extent and provides the first step in an effort to solve Maxwell's inhomogeneous equations.